A Rule-based System for Tuning Chord Progressions

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This article provides a set of heuristics with which to tune intervals, chords and chord progressions within a tonal setting. The frequency of each pitch, as it relates to tonic, is represented in the form $2^x * 3^y * 5^z$, $\{x, y, z\} \in Z$. Propositions of tuning and their resulting heuristics are supported by historical trends in music theory, as well as by experimental evidence. Detailed examples, including a short chorale by J.S. Bach, are presented as musical scores, Csound scores and MP3 sound files.

Overview

Tonality can be thought of as the marriage between periodic sounds, interwoven at multiple levels, and neuron firing patterns within the auditory system, similarly interwoven. Ideally, the former directly causes the latter with optimal efficiency.

Theoretical tuning systems restricted neither by the number of frequencies per octave nor by the accuracy of sound production may be systematically explored using digital sound synthesis techniques. Such tuning systems can target the human auditory system directly, taking into account the diverse interdependent physical, physiological and psychological factors affecting the perception of musical pitch.

The approach of tuning by ratios cannot be discounted out of hand. Just intonation can duplicate equal temperament to any desired level of precision. For example, to increase A-440 by a semitone in equal temperament, one would multiply 440 Hz by $\sqrt[12]{2}$. To an accuracy of seven decimal places (well beyond discrimination thresholds), this value would be A 440 x 1.0594631 = B-flat 466.1637640 Hz.

To reproduce this interval in just intonation, simply select a fraction whose numerator X and denominator Y provide the same approximation for $\sqrt[12]{2}$. In this case, X/Y = 10,594,631/10,000,000 = 1.0594631 which, when multiplied by 440 Hz, will yield exactly the same B-flat with a frequency of 466.1637640 Hz.

Any interval from equal temperament can be reproduced in this manner. One cent, approximately 1.0005778, can be represented as the fraction 10,005,778/10,000,000, and one hundredth of a cent, 1.0000058, as 10,000,058/10,000,000.

Just intonation can similarly approximate every interval from any system to arbitrary precision; thus, for all practical purposes, equal temperament, and indeed any tuning system, is a subset of just intonation. It is therefore not a question of whether just intonation, defined as a system in which all frequencies are related to each other as integer ratios, can provide the correct frequencies, but rather of which frequencies its precise fractions should produce.

The goal of attaining theoretical perfection with twelve notes has already been proven impossible:

> We want to adjust 36 connections independently (even more if we specify "just" sevenths or tritones) when there are only twelve things being connected; in fact, since the starting point is arbitrary, there are only 11 degrees of freedom. In these terms it is clear that there is absolutely no hope of our ever devising a truly just scale, one in which all intervals are in tune; in mathematical terms, it is a badly over-determined system, a non-trivial optimization problem. Since the people dealing with it have usually not been equipped with this viewpoint, it is no wonder that the problem has often been attacked intuitively rather than systematically, and that such a wide array of tunings and temperaments have been proposed as context-independent 'solutions'.
>
> — Donald E. Hall (1974)

The solution to finding a system, assuming it exists, in which any piece of tonal music can be made in tune, must then rely on using more than twelve tones. A truly modern method of tuning, which can only be implemented electronically, should determine the number and relative frequencies of pitches according to the capabilities of the human auditory system. The rules governing such a tuning system must be clear and their implementation consistent.

In order to demonstrate that tuning by ratios is practical in tonal music, two examples are tuned in complete detail using Csound. Please refer to the following section, "A Chord Progression Example", as well as "Appendix: Heuristic Analysis of a Bach Chorale", beginning on p. 22. Supplementary Csound orchestra and score files, along with mp3 sound files, can be found at:

https://Bobby1956.github.io/Rule-Based-Tuning/

A Chord Progression Example

The traditionally notated score in Figure 1 illustrates the familiar chord progression I-vi-ii-V-I. Beneath each note is a fraction representing the frequency relative to tonic. To obtain the frequencies in Hertz, simply multiply each such fraction by 65.406, the octave equivalent of C2. The rules by which to obtain the fractions representing relative frequencies are given in "Heuristics for Tuning Chords", starting on p. 15.

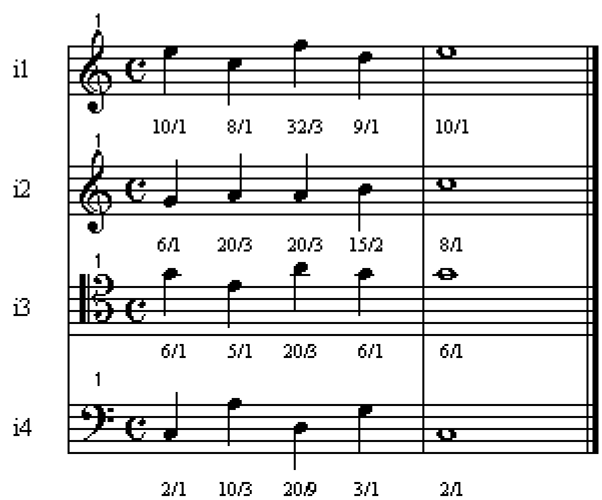

Figure 1. Chord Progression Example
https://Bobby1956.github.io/Rule-Based-Tuning/chrd_prog.html

Notice that the low D in the bass of measure one, beat three, is tuned as 20/9. In the same measure but in the treble and on beat four, there is another D tuned as 9/1, which is not an octave equivalent of 20/9. Such redefined notes are known as mutable tones. Mutable tones do not signify a weakness in just intonation; on the contrary, they indicate the already proven need for more than twelve tones, even in a simple and common two-chord progression such as ii-V.

Five-limit Tuning Compatible with Experimental Models

Helmholtz on Dissonance

Preeminent nineteenth-century scientist and physician Helmholtz described resonance not in terms of string lengths or tensions, but rather in physiological terms. According to Helmholtz, hair cells of the inner ear act as tuned resonators. His theory states that dissonance is caused by the beating of partials contained in two or more primary tones. Conversely, consonance results when two tones occur simultaneously without beating, as when two sine waves form the intervals of octaves, perfect fifths, perfect fourths, and, under certain circumstances, smaller intervals. According to Helmholtz:

> Combinational tones are the most general cause of beats. They are the sole cause of beats for simple tones which lie as much as, or more than, a minor third apart.
>
> — Hermann L.F. von Helmholtz (1954/1863)

The major scale proposed by Helmholtz uses intervals that are consistent with the theory of 5-limit just intonation (Figure 2). Another version of this scale proposed earlier by Rameau (1971/1722), also consistent with 5-limit theory, differs only by the second scale degree.

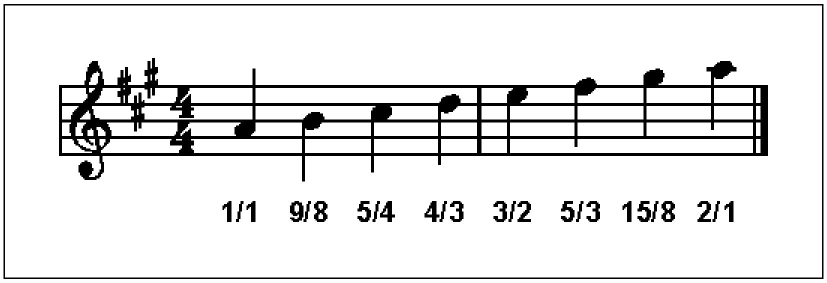

Figure 2. Major scale tuned according to Helmholtz (1954/1863)

Critical Bandwidth Theory

A breakthrough study (Plomp and Levelt 1965) demonstrated beyond doubt that partials sharing a critical band, an interval of roughly a minor third, interfere with each other (Figure 3). An important conclusion of this study is that, in general, the highest two consecutive partials not sharing a critical band are the fifth and sixth. Calculations revealed that when one six-partial sawtooth wave is gradually separated in frequency from the other, the only consonant intervals within the first octave are 5:6, 4:5, 3:4, 2:3, and 3:5, all basic ratios of 5-limit theory (Figure 4).

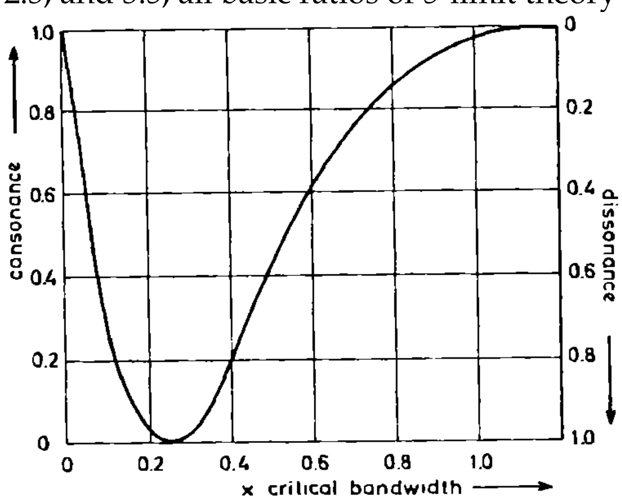

Figure 3. Standard consonance curve for two simple tones (Plomp and Levelt 1965)

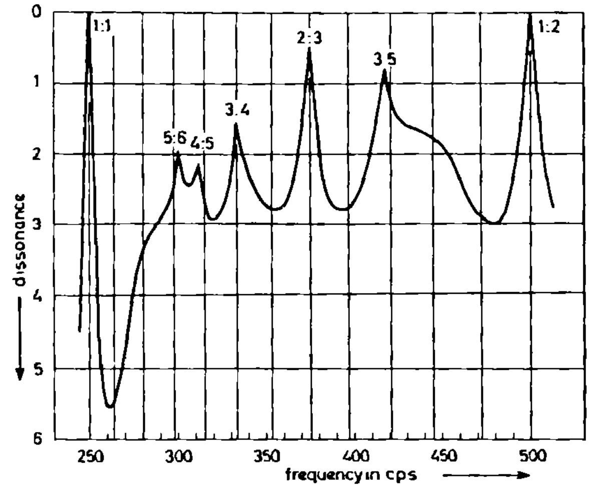

Figure 4. Computed dissonance for intervals between two six-harmonic tones, the lower of which is 250 Hz (Plomp and Levelt 1965)

Other Supporting Evidence for 5-limit Theory

A wealth of additional psychoacoustical research points to the first 5-6 partials as the spectral region of a periodic tone most closely associated with pitch perception. Research on the dominance region reveals that partials 3-5 are the principal determinants of pitch for a harmonic complex, and easily mask their higher partial neighbors (Ritsma and Engel 1967). There may be little or no phase locking to weak components which are close in frequency to stronger ones, and such weak components are likely masked from the overall time pattern of nerve impulses (Moore 1997).

Virtual pitch theory, based on the concept that a learning matrix of integer relationships is formed during the early stages of speech acquisition, asserts that both pitches and virtual pitches are inputs to a central processor that performs analysis of incoming signals (Terhardt 1974). Discrete partials are thus grouped into increasingly higher order perceptual units, pattern-matched against the learning matrix, and eventually interpreted as individual pitches. It logically follows that the same central processor is involved in grouping pitches into the higher order units we recognize as chords and chord progressions. Terhardt proposed that the third through sixth harmonics are dominant in the pitch identification process.

Harmonic Coincidence

Harmonic coincidence refers to the percentage of harmonics of a tone that coincide with harmonics from another tone. For two frequencies related as integers p and q, p and q being mutually prime, the coinciding harmonics will be multiples of pq. As an example, for the interval of an octave, p would be 1 and q would be 2. The coinciding harmonics for a sawtooth wave would be every other harmonic for p, and every harmonic of q. For a perfect fifth using the sawtooth timbre, with p = 2 and q = 3, the coinciding harmonics, having relative frequencies of {6, 12, 18, . . .}, would include every third harmonic of p and every other harmonic of q.

For each interval in Table 1, the number of coinciding harmonics is given. For brevity, each numerator and denominator is less than 9. Intervals generally associated with consonance tend to have a high number of coinciding harmonics, especially when the intervals are relatively close. Notable consonant exceptions are 8/1, the triple octave; 8/3, the perfect fourth plus octave; and 8/5, the minor sixth.

Notice that other than these three exceptions, the only intervals having no coinciding harmonics are those containing a seven in either the numerator or denominator. In general and without exception, no interval whose numerator or denominator contains a prime factor greater than five can have a coinciding harmonic within the first six harmonics, the "sweet spot" for the perception of periodic sounds.

A study of harmonic coincidence indicates that consonant intervals are those that share common lower harmonics (Vos and Vianen 1985). If harmonic coincidence could be proven essential for the perception of tonality, then intervals from which tonal chords are assembled would be limited to the form $2^x * 3^y * 5^z$, $\{x, y, z\} \in Z$.

Table 1. Coinciding harmonics for selected intervals using 6-partial sawtooth waves

INTERVAL	COINCIDING HARMONICS
1/1 Unison	6
2/1 Octave	3
3/1 Perfect Fifth Plus Octave	2
3/2 Perfect Fifth	2
4/1 Double Octave	1
4/3 Perfect Fourth	1
5/1 Major Third + 2 Octaves	1
5/2 Major Third + Octave	1
5/3 Major Sixth	1
5/4 Major Third	1
6/1 Perfect Fifth + 2 Octaves	1
6/5 Minor Third	1
7/1 (non-standard)	0
7/2 (non-standard)	0
7/3 (non-standard)	0
7/4 (non-standard)	0
7/5 (non-standard)	0
7/6 (non-standard)	0
8/1 Triple Octave	0
8/3 Perfect Fourth + Octave	0
8/5 Minor Sixth	0
8/7 (non-standard)	0

Listening Experiments Involving Tonal Chord Progressions

A study by the author examined the interplay of tuning system, timbre and duration. For each progression length, ranging in number from 1 - 7 chords, the most commonly occurring chord progression from 375 Bach chorales was electronically selected. Each progression was played 12 times. Variable parameters were tuning system (E.T., Pythagorean, Just), speed (0.5 sec. per chord, 1 sec. per chord), and timbre (sine wave, 8-partial sawtooth). Results indicated that equal temperament was favored when short durations and/or complex waveforms were used, while 5-limit just intonation was preferred when long durations and/or sine waves were employed. Equal temperament was marginally favored overall. Pythagorean tuning was simply not competitive with the other two tuning systems.

Future listening experiments based upon chords and chord progressions should explore the effects of varying such parameters as tuning system, timbre, duration, amplitude, range and chord spacing, all of which affect the perception of consonance. Of special interest are various timbres employing the first six partials.

Propositions of Five Limit Tuning

This section provides a list of general propositions upon which a series of heuristics, some of which are absolute and others that depend upon varying parameters, can be based. An illustration of an absolute heuristic is that of always tuning the perfect fifth as 3:2. Any close approximation will be perceived as the pure interval 3:2; conversely, any attempt at using the perfect fifth to fit some other relationship will sound too dissonant to be included in tonal harmony. The following list of propositions can be used to ensure that the principles of just intonation are maintained for most simple chord progressions.

Intervals

1. *Within a single set of simultaneously occurring frequencies, an octave equivalent always bears a relationship of* $2^n:1$.

This is taken as a given. All traditional Western tuning systems, including just intonation, Pythagorean tuning, and equal temperament, steadfastly obey this rule.

2. *Any tone in any chord can be transposed by any number of octaves, and the tuning, except for octave transposition, of the chord's individual notes will remain unchanged.*

There is no proof, but common practice shows that tunings are stable upon the octave transposition of individual tones. Any chord can be inverted to produce any other inversion, and the resulting tuning for this inversion, except for octave transposition of the affected notes, will remain unchanged.

> If one sound forms a perfect consonance with the fundamental sound, it will also form a perfect consonance with its octave; if another forms an imperfect consonance or a dissonance on the one hand, it will also form an imperfect consonance or a dissonance on the other; if another has to ascend or descend on the one hand, it will ascend or descend on the other, finally, everything that harmonizes on the one hand will also harmonize on the other.
>
> — Jean-Philippe Rameau (1971/1722)

3. *Within a single chord, the perfect fifth (or its octave equivalent) is always tuned in the proportion of 3:2 (or its octave equivalent).*

The perfect fifth can be derived from the third partial of the harmonic series. It is further the basis of all ancient Greek tuning systems, including Pythagorean tuning and its descendent, equal temperament. Rameau pointed out that nearly every chord in tonal music contains a perfect fifth (or its inverse, the perfect fourth, or one of their octave equivalents), and believed it to have only one tuning.

4. *Within a single chord, the perfect fourth (or its octave equivalent) is always tuned in the proportion of 4:3 (or its octave equivalent).*

The perfect fourth, as with its inversion the perfect fifth, will sound mistuned using any other ratio. It is such a basic tonal cornerstone that there should not be any exceptions within a single chord to this simple rule.

5. *Within a single chord, the major third (or its octave equivalent) is always tuned in the proportion of 5:4 (or its octave equivalent).*

As with the perfect fifth (3:2) and the perfect fourth (4:3), the major third, no matter how it is tuned, will be perceived as a single interval, in this case, 5:4. Mersenne, Rameau, Helmholtz and a host of other notable tuning specialists list only one tuning for this interval. By combining this proposition with propositions 3 and 4, many chords can quickly either be tuned or shown to be contradictory.

Chords

6. *Any valid chord can be transformed into another valid chord by dividing 1 by each of its relative frequencies.*

For example, the minor triad is 1 divided by the major triad. The ii$^{\varnothing 7}$ variety of the half-diminished seventh chord (135:160:192:240) is 1 divided by the dominant seventh chord.

7. *Any major triad must be tuned so that its relative frequencies are octave equivalents of 1:3:5.*

This is the fundamental tuning for the simplest and most common chord type. The components of the major triad are found in the first five partials of the harmonic series, with relative frequencies of 1, 2, 3, 4 and 5. When octave equivalents are removed from these first five partials, the values 1, 3 and 5 are left. (Closed root position would be 4:5:6.) For example, in the Table of Intervals for the Major Triad (Table 2), the first C in the left hand column, C1, is multiplied by 5:4 to obtain an E 5:4. This ratio corresponds to the interval of a just major third.

	C (1)	E (5:4)	G (3:2)
C (1)	1	5:4	3:2
E (5:4)	4:5	1	6:5
G (3:2)	2:3	5:6	1

Table 2. Table of intervals for the major triad

8. *Any minor triad must be tuned so that its relative frequencies are octave equivalents of 1:2, 1:3 and 1:5.*

The minor triad is regarded as a mirror image of the major triad. A detailed comparison between the intervals found in the major triad (Table 2) and the corresponding intervals found in the minor triad (Table 3) reveals that perfect fifths, major thirds and minor thirds are identical. The minor triad can thus be tuned as octave equivalents of 1, 1:3, 1:5, or equivalently, 15:5:3, or in closed root position, 10:12:15. If root C of a minor triad is taken as 1, then E-flat = 6:5 and G = 3:2.

	C (1)	E-flat (6:5)	G (3:2)
C (1)	1	6:5	3:2
E-flat (6:5)	5:6	1	5:4
G (3:2)	2:3	4:5	1

Table 3. Table of intervals for the minor triad

9. *Any dominant seventh chord must be tuned so that the frequency for each of its notes is related to its root as an octave equivalent of 2, 3, 5 and 1:9. (In closed root position, this will be 36:45:54:64.)*

This tuning is not a familiar one in the literature. It is completely avoided by Rameau, for instance, who favors the proportions 20:25:30:36. There is much to say for the seventh of a dominant seventh chord being related to its root as an octave equivalent of 1:9 (and to its tonic as an octave equivalent of 1:3). In the key of C, for example, there is no reason to change the tuning for the common tone F in extremely familiar progressions such as ii^6 – V^7 – I and IV – V^7 – I. Rameau's tuning for F relates to tonic C as an octave equivalent of 27:5, instead of the vastly more direct octave equivalent of 1:3. (Rameau's F, 27:5, is 21.5 cents sharper than the proposed common tone F, 16:3.) Had Rameau extended his own principle of the pre-eminent perfect fifth (and its inversion the perfect fourth) not only to individual chords but also to chord progressions, he would have found this much better tuning for the dominant seventh. More generally, he would have discovered that simple ratios, such as 1:3, serve as intermediaries to a greater number of chords and chord progressions than do their more complex counterparts, such as 27:5.

10. *Any major seventh chord must be tuned so that the frequencies for its notes are related to their root as octave equivalents of 2, 3, 5 and 15.*

This is in keeping with Rameau's practice of maintaining tunings of triads embedded within seventh chords. Contained within the major seventh chord is a major triad, to which is added the major seventh above the root. This major seventh acts as both the fifth of the major triad's major third, and the major third of the major triad's fifth. For a C major seventh chord, this tuning preserves the tunings of the embedded C major and E minor triads.

11. The function of chord x may be applied to another chord y by simply multiplying the relative frequencies of chord x by the fundamental frequency of chord y.

For example, if the frequency values for chord x, the dominant triad (12:15:18 in relation to tonic), are applied to chord y, in this case, tonic, then the values remain the same (12:15:18). If they are applied to a chord y in the case of a dominant, then each frequency of chord x is multiplied by the root of the dominant chord, an octave equivalent of 3:1, resulting in the values for the tonal function V/V, 36:45:54. The dominant of V/V, which is V/V/V, would similarly be tuned by multiplying the values for x, (12:15:18), by the root of y, 36, resulting in octave equivalents of 108:135:162.

12. The supertonic minor ii chord in a major key is tuned as octave equivalents of 10:9, 4:3 and 5:3.

For example, take tonic C as 1. We can assume the F, which is a cornerstone, being a fifth below tonic, to be related to C as 2:3. If a supertonic D ii chord, which serves the same pre-dominant function as IV, were to use the same subdominant tunings for both F and A, then the tuning for the ii chord would be D 10:9, F 4:3, and A 5:3.

13. The major IV chord is always tuned in relation to its local tonic as octave equivalents of 4:3, 5:3 and 2.

This tuning for major subdominant is rather self-evident, as by proposition 4 the root is known to be 4:3, and by proposition 7, the fixed tuning for the major triad is also known.

14. The dominant of a dominant of tonic (V/V/I) is tuned as octave equivalents of 9:8, 45:32 and 27:16.

In other words, the dominant of C 1 is G 3:2; the dominant of G 3:2, in turn, is 3:2 x 3:2 = D 9:4; the fifth of this secondary dominant D chord is 3:2 x 3:2 x 3:2 = 27:16.

Therefore, the tuning for this secondary dominant in relation to tonic C would be D 9:8, F# 45:32 and A 27:16.

15. The minor seventh chord is always tuned in the proportions 10:12:15:18.

This is in agreement with Rameau's axiom that any perfect fifth must have the relationship of 3:2, as is the case for the intervals 10:15 and 12:18. Additionally, the tuning for the minor triad 10:12:15 contained within the minor seventh maintains proposition 8. Similarly, the tuning for major triad 12:15:18, also found within the minor seventh chord, maintains proposition 7.

16. The major seventh chord is always tuned as relative frequencies of 8:10:12:15.

This tuning is again in agreement with Rameau's rule of tuning all perfect fifths in the ratio of 3:2. As with the minor seventh, the tunings of embedded major and minor triads within the larger chord are maintained.

Heuristics for Tuning Chords

Although the following heuristics will actually account for the majority of chords encountered within a clearly defined key area, they are only the beginning to a much broader picture of tonality. Chord progressions should be identified at a higher level than when simply analyzing the current sonority (Taube 1999). If a progression is stripped of its passing tones, for instance, and compared to another that never had them, the two could still be shown in many cases to be the same.

The following heuristics were based upon "Propositions of Five Limit Tuning", p. 10. Frequency values are based on tonic being an octave equivalent of 1. For detailed implementation, please refer to "Appendix: Heuristic Analysis of a Bach Chorale" on p. 22.

1. Tonic (I) major triad? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 5.
    c. The frequency of the fifth will be an octave equivalent of 3.

2. Tonic (i) minor triad? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 3/5.
    c. The frequency of the fifth will be an octave equivalent of 3.
3. Tonic major (I^7) chord? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 5.
    c. The frequency of the fifth will be an octave equivalent of 3.
    d. The frequency of the seventh will be an octave equivalent of 15.
4. Tonic minor (i^7) chord? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 3/5.
    c. The frequency of the fifth will be an octave equivalent of 3.
    d. The frequency of the seventh will be an octave equivalent of 9/5.
5. Supertonic (ii) minor triad? If yes,
    a. The frequency of the root will be an octave equivalent of 5/9.
    b. The frequency of the third will be an octave equivalent of 2/3.
    c. The frequency of the fifth will be an octave equivalent of 5/3.
6. Supertonic ii^7 minor seventh chord? If yes,
    a. The frequency of the root will be an octave equivalent of 5/9.
    b. The frequency of the third will be an octave equivalent of 2/3.
    c. The frequency of the fifth will be an octave equivalent of 5/3.
    d. The frequency of the seventh will be an octave equivalent of 2.
7. Supertonic half-diminished $ii^{\phi 7}$ chord? If yes,
    a. The frequency of the root will be an octave equivalent of 9/2.
    b. The frequency of the third will be an octave equivalent of 2/3.

   c. The frequency of the fifth will be an octave equivalent of 2/5.

   d. The frequency of the seventh will be an octave equivalent of 2.

8. Mediant (III/i) major triad? If yes,

   a. The frequency of the root will be an octave equivalent of 3/5.

   b. The frequency of the third will be an octave equivalent of 3.

   c. The frequency of the fifth will be an octave equivalent of 9/5.

9. Mediant (iii/I) minor triad? If yes,

   a. The frequency of the root will be an octave equivalent of 5.

   b. The frequency of the third will be an octave equivalent of 3.

   c. The frequency of the fifth will be an octave equivalent of 15.

10. Mediant (iii^7/I) minor seventh chord? If yes,

    a. The frequency of the root will be an octave equivalent of 5.

    b. The frequency of the third will be an octave equivalent of 3.

    c. The frequency of the fifth will be an octave equivalent of 15.

    d. The frequency of the seventh will be an octave equivalent of 9.

11. Subdominant (IV) major triad? If yes,

    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.

    b. The frequency of the third will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a subdominant.

    c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.

12. IV^7 major seventh chord? If yes,

    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.

    b. The frequency of the third will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a subdominant.

c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.

d. The frequency of the seventh will be an octave equivalent of 5 multiplied by the frequency of the root of the chord to which it is a subdominant.

13. Subdominant iv minor triad? If yes,

    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.

    b. The frequency of the third will be an octave equivalent of 2/5 multiplied by the frequency of the root of the chord to which it is a subdominant.

    c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.

14. iv^7 minor seventh chord? If yes,

    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.

    b. The frequency of the third will be an octave equivalent of 2/5 multiplied by the frequency of the root of the chord to which it is a subdominant.

    c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.

    d. The frequency of the seventh will be an octave equivalent of 3/5 multiplied by the frequency of the root of the chord to which it is a subdominant.

15. Dominant (V) triad? If yes,

    a. The frequency of the root will be an octave equivalent of 3 multiplied by the frequency of the root of the chord to which it is a dominant.

    b. The frequency of the third will be an octave equivalent of 15 multiplied by the frequency of the root of the chord to which it is a dominant.

    c. The frequency of the fifth will be an octave equivalent of 9 multiplied by the frequency of the root of the chord to which it is a dominant.

16. Dominant seventh (V^7) chord? If yes,
    a. The frequency of the root will be an octave equivalent of 3 multiplied by the frequency of the root of the chord to which it is a dominant.
    b. The frequency of the third will be an octave equivalent of 15 multiplied by the frequency of the root of the chord to which it is a dominant.
    c. The frequency of the fifth will be an octave equivalent of 9 multiplied by the frequency of the root of the chord to which it is a dominant.
    d. The frequency of the seventh will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a dominant.
17. Submediant (VI/i) major triad? If yes,
    a. The frequency of the root will be an octave equivalent of 2/5 multiplied by the frequency of the root of the chord to which it is a submediant.
    b. The frequency of the third will be an octave equivalent of 2/1 multiplied by the frequency of the root of the chord to which it is a submediant.
    c. The frequency of the fifth will be an octave equivalent of 3/5 multiplied by the frequency of the root of the chord to which it is a submediant.
18. Submediant (vi/I) minor triad? If yes,
    a. The frequency of the root will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a submediant.
    b. The frequency of the third will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a submediant.
    c. The frequency of the fifth will be an octave equivalent of 5 multiplied by the frequency of the root of the chord to which it is a submediant.
19. Submediant (vi^7) minor seventh chord? If yes,
    a. The frequency of the root will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a submediant.
    b. The frequency of the third will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a submediant.

c. The frequency of the fifth will be an octave equivalent of 5 multiplied by the frequency of the root of the chord to which it is a submediant.

d. The frequency of the seventh will be an octave equivalent of 3 multiplied by the frequency of the root of the chord to which it is a submediant.

20. Diminished leading vii^{o} triad? If yes,

    a. The frequency of the root will be an octave equivalent of 15 multiplied by the frequency of the root of the chord to which it is a leading vii^{o}.

    b. The frequency of the third will be an octave equivalent of 9 multiplied by the frequency of the root of the chord to which it is a leading vii^{o}.

    c. The frequency of the fifth will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a leading vii^{o}.

21. Half-diminished vii$^{\phi7}$ chord? If yes,

    a. CONTRADICTION. Root of ii and fifth of V create embedded syntonic comma.

22. For any octave equivalent of any perfect fifth within a single sonority, the lower tone will be related to the upper as an octave equivalent of 2:3.

23. For any octave equivalent for any perfect fourth within a single sonority, the frequency of the lower tone will be related to the frequency of the upper tone as 3:4.

24. For any octave equivalent of any major third within a single sonority, the lower tone will be related to the upper one as an octave equivalent of 4:5.

25. For any octave equivalent of a minor third serving as the third and fifth of a major triad, the third and fifth will be tuned, relative to each other, as octave equivalents of 5 and 3.

26. For any octave equivalent of a minor third serving as the root and third of a minor triad, the root and third will be tuned, relative to each other, as octave equivalents of 5 and 3.

27. For any octave equivalent of a minor third serving as the fifth and seventh of a dominant seventh chord, the fifth and seventh will be tuned, relative to each other, as octave equivalents of 27 and 32.

28. For any octave equivalent of any minor second within a single sonority, the lower tone will be related to the upper as an octave equivalent of 15:16.

29. For any octave equivalent of any major seventh within a single sonority, the lower tone will be related to the upper as an octave equivalent of 8:15.

Concluding Remarks

Clearly, the assertion that ratios are at the heart of traditional Western tuning systems is an irrefutable one. Music theorists should strive to produce, through rigorous listening experiments, the optimal tunings, expressed as precise ratios, for tonal chords and progressions.

Automating the process of tuning for entire pieces of tonal music is not at all straightforward. Pitch relationships can quickly become quite complex, resulting in myriad mutable tones. Tonal contradictions are common, especially in chromatic passages. Such contradictions do not constitute an indictment of just intonation, but rather indicate that passages containing such contradictions are not constructed within a strict theoretical framework.

The true value of just intonation is not that it can optimize the tunings for entire historical pieces of music, although it can more than occasionally do just that; rather, it is a framework, based upon the cause and effect relationships of periodic sounds upon the auditory system, within which to construct new pieces of music. Such rigorously constructed pieces could range in style from traditional classical music to modes of composition that have not been discovered. By carefully mapping and taking into account the limitations of the human auditory system, we can systematically explore the vast, uncharted regions of tonality.

Appendix: Heuristic Analysis of a Bach Chorale

This section is provided to demonstrate that entire pieces of music can be tuned according to precise rules of 5-limit just intonation, as given in "Heuristics for Tuning Chords", starting on p. 15. Detailed tunings are given for every vertical sonority contained in the Bach chorale, "Meine Seele erhebet den Herr" (Riemenschneider #130).

An MP3 sound file for this chorale, as well as the Csound orchestra and score files from which it was produced, is available at:

https://Bobby1956.github.io/Rule-Based-Tuning/bach1.html

Explanation of Tables

The following representation of a Bach chorale was initially created using intermediate output from the CONVERT program, one of three modules used by the author to catalogue chord progressions for 375 Bach Chorales. The CONVERT module is used to convert Csound scores into collections of discrete sonorities. Each line beginning with the letter 'i' is a Csound event representing a discrete sonority, or chord "snapshot", in which no voices enters, exits, repeats or changes pitch. The format for each event is as follows:

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7
Instrument number	Starting time	Duration	Note and octave (as given by The Acoustical Society of America)	Numerator of relative frequency	Denominator of relative frequency	Heuristic (See "Heuristics for Tuning Chords" on p. 15) and comments.

Riemenschneider #130
"Meine Seele erhebet den Herr"
Key: E Minor
1/1 = E2 (E2 = 82.407 cps)

Measure 1 Beat 1 Chord Type: i						
i1	0	2	B4	6	1	2.c
i2	0	2	E4	4	1	2.a
i3	0	2	G3	12	5	2.b
i4	0	2	E3	2	1	2.a

Measure 1 Beat 3 Chord Type: V^{6}/III						
i1	2	2	D5	36	5	15.a
i2	2	2	D4	18	5	15.a
i3	2	2	A3	27	10	15.c
i4	2	2	F#3	9	4	15.b
Measure 2 Beat 1 Chord Type: III						
i1	4	1	B4	6	1	8.b
i2	4	1	D4	18	5	8.c
i3	4	1	B3	3	1	8.b
i4	4	1	G3	12	5	8.a
Measure 2 Beat 2 Chord Type: $V^{4/3}$						
i1	5	1	B4	6	1	16.a
i2	5	1	D#4	15	4	16.b
i3	5	1	A3	8	3	16.d
i4	5	1	F#3	9	4	16.c
Measure 2 Beat 3 Chord Type: i						
i1	6	1	B4	6	1	2.c
i2	6	1	E4	4	1	2.a
i3	6	1	G3	12	5	2.b
i4	6	1	E3	2	1	2.a
Measure 2 Beat 4 Chord Type: V^{6}						
i1	7	1	B4	6	1	15.a
i2	7	1	F#4	9	2	15.c
i3	7	1	F#3	9	4	15.c
I4	7	1	D#3	15	8	15.b
Measure 3 Beat 1 Chord Type: VI^{6}						
i1	8	1	C5	32	5	17.a
i2	8	1	E4	4	1	17.b
i3	8	1	G3	12	5	17.c
i4	8	1	E3	2	1	17.b
Measure 3 Beat 2 Chord Type: $V^{6/5}$/III						
i1	9	1	C5	32	5	16.d
i2	9	1	D4	18	5	16.a
i3	9	1	A3	27	10	16.c
i4	9	1	F#3	9	4	16.b
Measure 3 Beat 3 Chord Type: III (+2)						
i1	10	1	B4	6	1	8.b
i2	10	1	D4	18	5	8.c
i3	10	1	A3	27	10	23(D->A)
i4	10	1	G3	12	5	8.a
Measure 3 Beat 4 Chord Type: i^{7}						
i1	11	1	B4	6	1	4.c
i2	11	1	D4	18	5	4.d
i3	11	1	G3	12	5	4.b
i4	11	1	E3	2	1	4.a
Measure 4 Beat 1 Chord Type: $iv^{6/5}$						
i1	12	1	A4	16	3	14.a
i2	12	1	E4	4	1	14.c
i3	12	1	G3	12	5	14.d
i4	12	1	C3	8	5	14.b
Measure 4 Beat 2 Chord Type: iv^{7} (-3)						
i1	13	1	A4	16	3	14.a
i2	13	1	E4	4	1	14.c
i3	13	1	G3	12	5	14.d
i4	13	1	A2	4	3	14.a
Measure 4 Beat 3 Chord Type: V/III (microtonal change in soprano)						
i1	14	2	A4	27	5	15.c
i2	14	2	D4	18	5	15.a
i3	14	2	F#3	9	4	15.b
I4	14	2	D3	9	5	15.a

Measure 5 Beat 1 Chord Type: III						
i1	16	4	G4	24	5	8.a
i2	16	4	D4	18	5	8.c
i3	16	4	B3	3	1	8.b
i4	16	4	G2	6	5	8.a
Measure 6 Beat 1 Chord Type: III						
i1	20	2	B4	6	1	8.b
i2	20	2	D4	18	5	8.c
i3	20	2	G3	12	5	8.a
i4	20	2	G3	12	5	8.a
Measure 6 Beat 3 Chord Type: III^6						
i1	22	2	D5	36	5	8.c
i2	22	2	D4	18	5	8.c
i3	22	2	G3	12	5	8.a
i4	22	2	B2	3	2	8.b
Measure 7 Beat 1 Chord Type: V/III						
i1	24	8	A4	27	5	15.c
i2	24	8	D4	18	5	15.a
i3	24	8	F#3	9	4	15.b
i4	24	8	D3	9	5	15.a
Measure 9 Beat 1 Chord Type: V/III						
i1	32	8	A4	27	5	15.c
i2	32	8	D4	18	5	15.a
i3	32	8	F#3	9	4	15.b
i4	32	8	D3	9	5	15.a
Measure 11 Beat 1 Chord Type: iv^6						
i1	40	2	E4	4	1	13.c
i2	40	2	E4	4	1	13.c
i3	40	2	A3	8	3	13.a
i4	40	2	C3	8	5	13.b
Measure 12 Beat 1 Chord Type: $i^{6/4}$						
i1	42	1	G4	24	5	2.b
i2	42	1	E4	4	1	2.a
i3	42	1	B3	3	1	2.c
i4	42	1	B2	3	2	2.c
Measure 12 Beat 2 Chord Type: $i^{6/4}$ (+4, 4 in bass)						
i1	43	1	G4	24	5	2.b
i2	43	1	E4	4	1	2.a
i3	43	1	B3	3	1	2.c
i4	43	1	A2	4	3	22(E->A)
Measure 12 Beat 3 Chord Type: V						
i1	44	1	F#4	9	2	15.c
i2	44	1	D#4	15	4	15.b
i3	44	1	B3	3	1	15.a
i4	44	1	B2	3	2	15.a
Measure 12 Beat 4 Chord Type: V^7						
i1	45	1	F#4	9	2	16.c
i2	45	1	D#4	15	4	16.b
i3	45	1	A3	8	3	16.d
i4	45	1	B2	3	2	16.a
Measure 13 Beat 1 Chord Type: I						
i1	46	4	E4	4	1	2.a
i2	46	4	B3	3	1	2.c
i3	46	4	G3	12	5	2.b
i4	46	4	E2	1	1	2.a

References

Hall, D. 1974. "Quantitative Evaluation of Musical Scale Tunings." *American Journal of Physics*, 42: 543-52.

Helmholtz, H. L. F. von. 1954. *On the Sensations of Tone as a Psychological basis for the Theory of Music*. New York: Dover. (Based on 4th German edition of 1877; original work published 1863)

Moore, B. C. J. 1997. *An Introduction to the Psychology of Hearing*. London: Academic Press.

Plomp, R. and Levelt, W. J. M. 1965. "Tonal Consonance and Critical Bandwidth." *Journal of the Acoustical Society of America*, 38: 548-60.

Rameau, J. 1971. *Treatise on Harmony*. New York: Dover. (Based on Paris edition of 1722)

Ritsma, R. and Engel, F. 1967. "Frequencies dominant in the perception of the pitch of complex sounds." *Journal of the Acoustical Society of America*, 42: 191-98.

Taube, H. 1999. "Automatic Tonal Analysis: Toward the Implementation of a Music Theory Workbench." *Computer Music Journal*, 23: 18-32.

Terhardt, E. 1974. "Pitch, Consonance and Harmony." *Journal of the Acoustical Society of America*, 55: 1061-69.

Vos, J. and van Vianen, B. G. 1985. "Thresholds for discrimination between pure and tempered intervals: the relevance of nearly coinciding harmonics." *Journal of the Acoustical Society of America*, 77: 176-187.